BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 2

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CONTENTS

I. REVIEW

Here's one more piece of terminology that will come in handy.

Definition 1. Let $\varphi_1 : U_1 \to \widehat{U_1}$ and $\varphi_2 : U_2 \to \widehat{U_2}$ be two complex charts on a topological surface X. Then (U_1, φ_1), (U_2, φ_2) are holomorphically compatible if either $U_1 \cap U_2 = \varnothing$, or the transition function

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is holomorphic.

Last time we:

(1) Reviewed/learned some facts about holomorphic functions. The takeaway is that holomorphic functions are analytic, meaning they are given by power series.
(2) Defined Riemann surfaces. The upshot is that a Riemann surface is something that locally looks like $\mathbb{C}$. More precisely, it is equipped with a holomorphic atlas, whose charts are holomorphically compatible, so the transition functions are holomorphic.

Bonus: here's a proof of Liouville's Theorem.

Proof. Suppose f is entire and bounded, so there exists $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Fix $z \in \mathbb{C}$ and for each $R \in \mathbb{R}_{>0}$, let γ_R be the circle $|\zeta - z| = R$ centered at z of radius R, traversed counterclockwise. By the generalized Cauchy integral formula, then

$$\begin{aligned} |f'(z)| &= \left| \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta \right| \leq \frac{1}{2\pi} \int_{\gamma_R} \frac{|f(\zeta)|}{|\zeta - z|^2} \, |d\zeta| = \frac{1}{2\pi} \int_{\gamma_R} \frac{M}{R^2} |d\zeta| \\ &= \frac{1}{2\pi} \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R} \,. \end{aligned}$$

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This holds for all $R \in \mathbb{R}_{>0}$, so taking $R \to \infty$ we have $|f'(z)| \leq \dfrac{M}{R} \to 0$. $\square$

II. Algebraic curves as Riemann surfaces

We'll be following chapter 1 of Miranda for today.

Example 2 (Graph of a holomorphic function)**.** Let $U \subseteq \mathbb{C}$ be a connected open subset, and let $f : U \to \mathbb{C}$ be holomorphic. Consider the graph of f

$$X = \{(z, f(z)) \in \mathbb{C}^2 : z \in U\}$$

equipped with the subspace topology. Let $\pi : X \to U$ be the projection $(z, w) \mapsto z$. Then π is a homeomorphism with inverse $z \mapsto (z, f(z))$. Thus π is a coordinate chart on X, and X is a Riemann surface, with an atlas consisting of the single chart (X, π). (Once we define morphisms of Riemann surfaces, we'll see that the graph is isomorphic to U.)

Similarly, given an finite collection $f_1, \ldots, f_m$ of holomorphic functions defined on U, their graph

$$X = \{(z, f_1(z), \ldots, f_m(z)) \in \mathbb{C}^{m+1} : z \in U\}$$

is a Riemann surface.

Definition 3. Let k be a field and $n \in \mathbb{Z}_{\geq 0}$. Affine n-space over k is defined as $\mathbb{A}^n = k^n$.

Definition 4. Given a polynomial $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$, define its vanishing locus or (zero locus) in $\mathbb{A}^n$ by

$$\mathbb{V}(f) := \{(x_1, \ldots, x_n) \in \mathbb{A}^n : f(x_1, \ldots, x_n) = 0\}.$$

Theorem 5. *If $f(x, y) \in \mathbb{C}[x, y]$ is irreducible, then its zero locus $\mathbb{V}(f)$ is connected.*

Remark 6. We won't prove this. It's not hard to show that it's connected in the Zariski topology, but considerably harder in the usual complex topology.

Definition 7. An affine plane curve X is the zero locus in $\mathbb{A}^2$ of an irreducible polynomial $f(x, y) \in \mathbb{C}[x, y]$. A point $P \in X$ is singular if $f_x(P) = f_y(P) = 0$; otherwise it is nonsingular. If all the points of X are nonsingular, then we say that X is nonsingular or smooth.

Remark 8. We often write $X : f(x, y) = 0$ to mean $X = \mathbb{V}(f)$.

To show that a smooth affine plane curve is a Riemann surface, we'll need the following result from analysis.

Theorem 9 (Implicit Function Theorem)**.** *Let $f(x, y) \in \mathbb{C}[x, y]$, let $X = \mathbb{V}(f)$ be its zero locus, and let $P = (x_0, y_0) \in X$. Suppose that $\dfrac{\partial f}{\partial y}(P) \neq 0$. Then there is a function $g(x)$ defined and holomorphic in a neighborhood of x_0 such that, in a neighborhood of P, X is equal to the graph $y = g(x)$, i.e., $f(x, g(x)) = 0$ for all x in a neighborhood of x_0.*

Remark 10. The analogous result is true if we instead assume $\dfrac{\partial f}{\partial x}(P) \neq 0$

Theorem 11. *Let $X = \mathbb{V}(f)$ be a nonsingular affine plane curve. Then X is a Riemann surface.*

Proof. The idea is to use the implicit function theorem to locally express X as the graph of a holomorphic function. As in the example above, this allows us to use the projection map to construct a chart around each point.

Given a point $P \in X$, then $f_x(P) \neq 0$ or $f_y(P) \neq 0$ since X is smooth. If $f_y(P) \neq 0$ then there is a open neighborhood U of P and a holomorphic function g such that $y = g(x)$ on this neighborhood. Thus we take (U, π_x) as a chart at P, where $\pi_x : (x, y) \mapsto x$. Similarly, if $f_x(P) \neq 0$ then there exists an open neighborhood V of P and a holomorphic function h such that $x = h(y)$ on V. In this case, we take (V, π_y) as our chart at P.

To see that these charts form a holomorphic atlas, note that

$$\pi_y \circ \pi_x^{-1} : z \mapsto (z, g(z)) \mapsto g(z)$$

which is holomorphic. □

Example 12.

(1) An (affine) hyperelliptic curve X is an affine plane curve given by an equation of the form $y^2 = f(x)$, where f is a polynomial with distinct roots and of degree $\deg(f) \geq 5$. If $\deg(f) = 3$ or 4, X is instead called an elliptic curve. (We will later see that, denoting the degree of f by $2g + 1$ or $2g + 2$, then the curve X has genus g.)

(2) An (affine) Fermat curve X is an affine plane curve given by an equation of the form

$$x^d + y^d = 1$$

for some $d \geq 1$. It is so called because nontrivial rational points on X would give counterexamples to Fermat's Last Theorem.

III. Projective plane curves

III.1. **The projective plane.**

Affine plane curves are not complete: they usually run off to infinity. Projective space provides the right setting in which to compactify them

Definition 13. Let k be a field. The projective plane over k is the set of 1-dimensional subspaces of k^3. Equivalently, define

$$\mathbb{P}^2 = \frac{k^3 \setminus \{(0,0,0)\}}{\sim}$$

where $\sim$ is the equivalence relation defined by: given $v \in k^3 \setminus \{(0,0,0)\}$, $v \sim \lambda v$ for all $\lambda \in \mathbb{C}^\times$. As before, we denote equivalence classes in $\mathbb{P}^2$ by $[z_0 : z_1 : z_2]$.We equip $\mathbb{P}^2$ with the quotient topology.

The entries of $[z_0 : z_1 : z_2]$ are called homogeneous coordinates. The homogeneous coordinates of a point in $\mathbb{P}^2$ are not unique, since we can always scale by an element of $k^\times$. However, whether a coordinate is zero or not is well-defined.

Remark 14. Note that $[0 : 0 : 0]$ is *not* a point in $\mathbb{P}^2$!

The projective plane $\mathbb{P}^2$ comes with a standard affine cover consisting of the three open subsets

$$U_j := \{[z_0 : z_1 : z_2] : z_j \neq 0\}$$

for $j = 0, 1, 2$. Each open set U_j is homeomorphic to the affine plane $\mathbb{A}^2$. For instance, the homeomorphism for U_0 is given by

$$U_0 \to \mathbb{A}^2$$
$$[z_0 : z_1 : z_2] = [1 : z_1/z_0 : z_2/z_0] \mapsto \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right),$$

with inverse

$$\mathbb{A}^2 \to U_0$$
$$(x, y) \mapsto [1 : x : y].$$

Remark 15. A subset $S \subseteq \mathbb{P}^2$ is open iff $S \cap U_j$ is open for all $j = 0, 1, 2$.

We now restrict to the case of $k = \mathbb{C}$.

Lemma 1. $\mathbb{P}^2$ *is compact.*

Proof. Note that every point $[z_0 : z_1 : z_2]$ in $\mathbb{P}^2$ has a representative with

$$|z_0| \leq 1, |z_1| \leq 1, |z_2| \leq 1.$$

Thus $\mathbb{P}^2$ is the image of the compact set

$$\{(z_0, z_1, z_2) \in \mathbb{C}^2 : |z_j| \leq 1 \;\forall j = 0, 1, 2\}$$

under the quotient map $\mathbb{C}^3 \to \mathbb{P}^2$. □

III.2. **Projective plane curves.** Now that we have a compact ambient space to work in, we'd like to define curves in the projective plane. The fact that the value of a homogeneous coordinate is not well-defined creates some restrictions.

Definition 16. A polynomial $F \in \mathbb{C}[x, y, z]$ is homogeneous if every monomial term has the same total degree. This total degree is called the degree of F.

Example 17. The polynomial $F = y^2z + 2xyz - x^3 - xz^2$ is homogeneous of degree 3.

Let $F \in \mathbb{C}[x, y, z]$ be homogeneous of degree d. As suggested above, it doesn't make sense to evaluate a point $[a : b : c] \in \mathbb{P}^2$. Indeed, we have $[a : b : c] = [\lambda a : \lambda b : \lambda c]$ for every $\lambda \in \mathbb{C}^\times$, but

$$F(\lambda a, \lambda b, \lambda c) = \lambda^d F(a, b, c)$$

since F is homogeneous of degree d. However, it does make sense to ask whether $F(a, b, c)$ is zero or not, since this is preserved by scaling by a nonzero scalar.

Definition 18. Let $F \in \mathbb{C}[x, y, z]$ be a homogeneous polynomial. The vanshing locus or zero locus of F is

$$\mathbb{V}(F) := \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 : F(x_0, x_1, x_2) = 0\}$$

Lemma 2. $\mathbb{V}(F)$ *is a closed subset of* $\mathbb{P}^2$.

The intersection X_j of X with the open subsets U_j is an affine plane curve when mapped under the homeomorphism $U_j \to \mathbb{A}^2$. For example, on U_0 where $x_0 \neq 0$, we have

$$X_0 = X \cap U_0 \xrightarrow{\sim} \{(u, v) \in \mathbb{A}^2 : F(1, u, v)\}$$
$$[x_0 : x_1 : x_2] = [1 : x_1/x_0 : x_2/x_0] \mapsto (x_1/x_0, x_2/x_0)$$

$F(1, u, v)$ is called the dehomogenization of F with respect to the variable x_0, and is denoted by F_*. Letting $f = F_*$, so $f(u, v) = F(1, u, v)$, then X_0 is given by $f(u, v) = 0$.

Definition 19. Let $X : F(x, y, z) = 0$ be a projective plane curve. A point PX is singular if $F_x(P) = F_y(P) = F_z(P) = 0$; otherwise it is nonsingular. If all the points of X are nonsingular, then X is nonsingular or smooth.

Remark 20. In other words, $X = \mathbb{V}(F)$ is nonsingular iff there are no common solutions to the system of equations

$$F = F_x = F_y = F_z = 0$$

in $\mathbb{P}^2$.

Nonsingularity is a local property, hence can be checked on the standard affine open cover.

Proposition 21. *Let $X : F(x, y, z) = 0$ be a projective plane curve. Then X is nonsingular iff X_j is nonsingular for each $j = 0, 1, 2$.*

Proof. HW? It's mostly straightforward, except at one point one uses Euler's formula for homogeneous polynomials. □

Proposition 22. *Let $X : F(x_0, x_1, x_2) = 0$ be a nonsingular projective plane curve, where $F \in \mathbb{C}[x_0, x_1, x_2]$ is homogeneous. Then X is a compact, connected Riemann surface. Moreover, at every point of X one can take a ratio of the homogeneous coordinates as a local coordinate.*

Proof. We give just a sketch. First, note that X is a closed subset of the compact set $\mathbb{P}^2$, hence is itself compact. One can show that a nonsingular homogeneous polynomial is automatically irreducible, but we won't prove this.

Here's a sketch of the rest of the proof. Recall that the three open subsets X_0, X_1, X_2 are smooth affine plane curves, hence are Riemann surfaces by our results above. Or, more precisely, letting Y_0, Y_1, Y_2 be their respective images under the homeomorphisms $U_j \xrightarrow{\sim} \mathbb{A}^2$, then Y_0, Y_1, Y_2 are smooth affine plane curves. Recall that the coordinate charts on the Y_j are simply one of the projections onto a coordinate. Consider X_0, for example. A coordinate map on Y_0 is given by one of the projections, i.e., it is either of the form

$$(z_1, z_2) \mapsto z_1 \quad \text{or} \quad (z_1, z_2) \mapsto z_2$$

Composing with the homeomorphism

$$U_0 \xrightarrow{\sim} \mathbb{A}^2$$
$$[x_0 : x_1 : x_2] \mapsto (x_1/x_0, x_2/x_0)$$

we find that the coordinate maps of X_0 are of the form

$$[x_0 : x_1 : x_2] \mapsto x_1/x_0 \quad \text{or} \quad [x_0 : x_1 : x_2] \mapsto x_2/x_0\,.$$

Every point $P \in X$ is contained in some X_j, so to show that the union of the atlases of X_0, X_1, X_2 yields an atlas for X, it suffices to check that the charts on X_i and X_j are holomorphically compatible for all i, j. For example, consider a point $P \in X$ that is in both X_0 and X_1. Then $P = [a_0, a_1, a_2]$ where $a_0 \neq 0$ and $a_1 \neq 0$. Suppose the coordinate of X_0 near P is

$$\begin{aligned} \psi_0 : X_0 &\to Y_0 \to \mathbb{C} \\ [x_0 : x_1 : x_2] &\mapsto (x_1/x_0, x_2/x_0) \mapsto x_1/x_0 \end{aligned}$$

with inverse

$$\begin{aligned} \psi_0^{-1} : \mathbb{C} &\to Y_0 \to X_0 \\ w &\mapsto (w, h(w)) \mapsto [1 : w : h(w)] \end{aligned}$$

for some holomorphic complex-valued function h, and the coordinate of X_1 near P is

$$\begin{aligned} \psi_1 : X_1 &\to Y_1 \to \mathbb{C} \\ [x_0 : x_1 : x_2] &\mapsto (x_0/x_1, x_2/x_1) \mapsto x_2/x_1 \,. \end{aligned}$$

Then the transition function is given by

$$\psi_1 \circ \psi_0^{-1} : w \mapsto [1 : w : h(w)] \mapsto h(w)/w$$

and this is holomorphic since $w \neq 0$, as the point $[1 : w : h(w)]$ is in X_1.

One can similarly check that all the other possible combinations of charts also yield holomorphic transition functions. Thus all the charts of X_0, X_1, X_2 are holomorphically compatible, hence form an atlas for X. □